Anti-Commutator Theorem and Essential Self-Adjointness

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Abstract. In this article, the self-adjoint extensions of symmetric operators satisfying anti-commutation relations are considered. Then it is proven that an anti-commutative type of the Glimm-Jaffe-Nelson commutator theorem follows. In addition we investigate its application to the Dirac operator with external potentials.

1 Introduction and Main Theorem

In this paper we consider the essential self-adjointness of symmetric operators satisfying anti-commutation relations. Let H be a linear operator on a separable Hilbert space $\mathcal{H}$. It is said that H is symmetric if $H \subset H^*$, that is $\mathcal{D}(H) \subset \mathcal{D}(H^*)$ and $H^*\Psi = H\Psi$ for $\Psi \in \mathcal{D}(H)$, and we say that H is self-adjoint if $H = H^*$. In particular it is said that H is essentially self-adjoint if its closure $\overline{H}$ is self-adjoint. We are interested in conditions under which a symmetric operator has its self-adjoint extension. The Glimm-Jaffe-Nelson commutator theorem ([1]; Theorem 2.32, [7]; Theorem X.36) is one criteria to show the essential self-adjointness for commuting symmetric operators. The commutator theorem shows that if a symmetric operator H and a self-adjoint operator S obey a commutation relation on a dense subspace $\mathcal{D}$ which is a core of S, then H is essentially self-adjoint on $\mathcal{D}$. Historically, Glimm-Jaffe [3] and Nelson [4] investigated the commutator theorem for the application to quantum field models, and Faris-Lavine [2] considered its application to the quantum mechanical models. We investigate an anti-commutative version of the commutator theorem.

Let us consider symmetric operators X and Y. Then the real part and the imaginary part of the inner product $(X\Psi, Y\Psi)$ for $\Psi \in \mathcal{D}(XY) \cap \mathcal{D}(YX)$ are expressed by

$$\mathrm{Re}\,(X\Psi, Y\Psi) \;=\; \frac{1}{2}(\Psi, \{X,Y\}\Psi) \;, \tag{1}$$

and

$$\mathrm{Im}\,(X\Psi, Y\Psi) \;=\; \frac{1}{2}(\Psi, [X,Y]\Psi) \;, \tag{2}$$

where $\{X,Y\} = XY + YX$ and $[X,Y] = XY - YX$. In the commutator theorem, the imaginary part (2) is estimated. Instead of this, we estimate the real part (1), and obtain an anti-commutative version of the commutator theorem. Here we overview the commutator theorem, and later we state the main theorem.

Let H and O be linear operators on $\mathcal{H}$. We introduce the following conditions.

(C.1) H is symmetric and O is self-adjoint.
(C.2) There exists $\delta_0 > 0$ such that for $\Psi \in \mathcal{D}(O)$,

$$\delta_0(\Psi,\Psi) \;\leq\; |(\Psi, O\Psi)|.$$

(C.3) O has a core $\mathcal{D}_0$ satisfying $\mathcal{D}_0 \subset \mathcal{D}(H)$, and there exist constants $a \geq 0$ and $b \geq 0$ such that for all $\Psi \in \mathcal{D}_0$,

$$\|H\Psi\| \;\leq\; a\|O\Psi\| + b\|\Psi\|.$$

Proposition A (Glimm-Jaffe-Nelson Commutator Theorem)
Let H and O be the operators satisfying **(C.1)-(C.3)**. In addition we suppose **(i)** or **(ii)** below :
(i) There exists a constant $c_+ > 0$ such that for all $\Psi \in \mathcal{D}_0$,

$$\Big| (H\Psi, O\Psi) - (O\Psi, H\Psi) \Big| \;\leq\; c_+ \, |(\Psi, O\Psi)|. \tag{3}$$

(ii) There exists a constant $c_- > 0$ such that for all $\Psi \in \mathcal{D}_0$,

$$c_- \, |(\Psi, O\Psi)| \;\leq\; \Big| (H\Psi, O\Psi) - (O\Psi, H\Psi) \Big| \, . \tag{4}$$

Then H is essentially self-adjoint on $\mathcal{D}_0$.

Remark 1 *In the commutator theorem, the condition **(i)** is usually supposed. But we will see that the commutator theorem also follows under the condition **(ii)**.*

(Proof of Proposition A) Since the proof with the condition **(i)** is well-known, let us only give the proof by supposing **(ii)**. From the general theorem ([7], Theorem X.1), it is enough to show that for some $z \in \mathbf{C}\backslash\mathbf{R}$, $\dim\ker\big((H_{\lceil\mathcal{D}_0})^* + z^\sharp \big) = 0$ where $z^\sharp = z, z^*$. Since $\mathcal{D}_0$ is a core of O, it is seen from **(C.3)** and **(ii)** that $\mathcal{D}(O) \subset \mathcal{D}(\overline{H_{\lceil\mathcal{D}_0}})$ and for $\Phi \in \mathcal{D}(O)$,

$$c_- \, |(\Phi, O\Phi)| \;\leq\; \Big| (\overline{H_{\lceil\mathcal{D}_0}}\Phi, O\Phi) - (O\Phi, \overline{H_{\lceil\mathcal{D}_0}}\Phi) \Big|. \tag{5}$$

Let $\Psi \in \mathcal{D}((H_{\lceil\mathcal{D}_0})^*)$ and $\Xi = O^{-1}\Psi \in \mathcal{D}(O)$. Here it follows from **(C.2)** that O^{-1} is bounded. By using $((H_{\lceil\mathcal{D}_0})^*)^* = \overline{H_{\lceil\mathcal{D}_0}}$, we obtain that

$$\mathrm{Im}\Big(\Xi, ((H_{\lceil\mathcal{D}_0})^* + z^\sharp)\Psi\Big) \;=\; \frac{1}{2}\Big((\overline{H_{\lceil\mathcal{D}_0}}\Xi, O\Xi) - (O\Xi, \overline{H_{\lceil\mathcal{D}_0}}\Xi)\Big) \;+\; \mathrm{Im}z^\sharp\,(\Xi, O\Xi). \tag{6}$$

Then for $z \in \mathbf{C}\backslash\mathbf{R}$ satisfying $|\mathrm{Im}z| < \frac{c_-}{2}$, we have from (5) and (6), that

$$\Big|\mathrm{Im}\,\Big(\Xi, ((H_{\lceil\mathcal{D}_0})^* + z^\sharp)\Psi\Big)\Big| \;\geq\; \left(\frac{c_-}{2} - |\mathrm{Im}z|\right)|(\Xi, O\Xi)| \geq \delta_O\left(\frac{c_-}{2} - |\mathrm{Im}z|\right)(\Xi,\Xi). \tag{7}$$

Hence, if $\Psi \in \ker\big((H_{\lceil\mathcal{D}_0})^* + z^\sharp\big)$, we have $\Xi = O^{-1}\Psi = 0$ from (7), and then $\Psi = 0$. Thus the proof is obtained. ■

Here we state the main theorem.

Theorem 1 Assume **(C.1)-(C.3)**. In addition we suppose the following **(I)** or **(II)** holds.
(I) There exists a constant $d_+ > 0$ such that for all $\Psi \in \mathcal{D}_0$,

$$\left| (H\Psi, O\Psi) + (O\Psi, H\Psi) \right| \leq d_+ \left|(\Psi, O\Psi)\right|. \tag{8}$$

(II) There exists a constant $d_- > 0$ such that for all $\Psi \in \mathcal{D}_0$,

$$d_- \left|(\Psi, O\Psi)\right| \leq \left| (H\Psi, O\Psi) + (O\Psi, H\Psi) \right|. \tag{9}$$

Then H is essentially self-adjoint on $\mathcal{D}_0$.

(Proof of Theorem 1) In a similar way to Proposition A, let us show that for some $z \in \mathbf{C} \backslash \mathbf{R}$, $\dim \ker \left((H_{\lceil \mathcal{D}_0})^* + z^\sharp \right) = 0$ where $z^\sharp = z,\ z^*$. Let $\Psi \in \mathcal{D}((H_{\lceil \mathcal{D}_0})^*)$ and $\Xi = O^{-1}\Psi \in \mathcal{D}(O)$. Then by $((H_{\lceil \mathcal{D}_0})^*)^* = \overline{H_{\lceil \mathcal{D}_0}}$, we see that

$$\mathrm{Re}\left(\Xi, ((H_{\lceil \mathcal{D}_0})^* + z^\sharp)\Psi\right) = \frac{1}{2}\left((\overline{H_{\lceil \mathcal{D}_0}}\Xi, A\Xi) + (O\Xi, \overline{H_{\lceil \mathcal{D}_0}}\Xi) \right) + \mathrm{Re}z\,(\Xi, O\Xi). \tag{10}$$

First we assume that **(I)** holds. Let us take $z \in \mathbf{C} \backslash \mathbf{R}$ satisfying $|\mathrm{Re} z| > \frac{d_+}{2}$. Since $\mathcal{D}_0$ is a core of O, it follows from **(C.3)** and **(I)** that $\mathcal{D}(O) \subset \mathcal{D}(\overline{H_{\lceil \mathcal{D}_0}})$ and for $\Phi \in \mathcal{D}(O)$,

$$\left| (\overline{H_{\lceil \mathcal{D}_0}}\Phi, O\Phi) + (O\Phi, \overline{H_{\lceil \mathcal{D}_0}}\Phi) \right| \leq d_+ \left|(\Phi, O\Phi)\right|. \tag{11}$$

By (10) and (11), we have

$$\left|\mathrm{Re}\left(\Xi, ((H_{\lceil \mathcal{D}_0})^* + z^\sharp)\Psi\right)\right| \geq \left(|\mathrm{Re} z| - \frac{d_+}{2}\right)|(\Xi, O\Xi)| \geq \delta_0 \left(|\mathrm{Re} z| - \frac{d_+}{2}\right)(\Xi, \Xi). \tag{12}$$

Hence if $\Psi \in \ker\left((H_{\lceil \mathcal{D}_0})^* + z^\sharp\right)$, we have $\Xi = O^{-1}\Psi = 0$ from (12). Then we have $\Psi = 0$. Next we suppose that **(II)** follows. Let us take $z \in \mathbf{C} \backslash \mathbf{R}$ satisfying $|\mathrm{Re} z| < \frac{d_-}{2}$. Since $\mathcal{D}_0$ is a core of O, it also follows from **(C.3)** and **(II)** that $\mathcal{D}(O) \subset \mathcal{D}(\overline{H_{\lceil \mathcal{D}_0}})$ and for $\Phi \in \mathcal{D}(O)$,

$$d_- \left|(\Phi, O\Phi)\right| \leq \left| (\overline{H_{\lceil \mathcal{D}_0}}\Phi, O\Phi) + (O\Psi, \overline{H_{\lceil \mathcal{D}_0}}\Phi) \right|. \tag{13}$$

Then from (10) and (13), we have

$$\left|\mathrm{Re}\left(\Xi, ((H_{\lceil \mathcal{D}_0})^* + z^\sharp)\Psi\right)\right| \geq \left(\frac{d_-}{2} - |\mathrm{Re} z|\right)|(\Xi, O\Xi)| \geq \delta_0 \left(\frac{d_-}{2} - |\mathrm{Re} z|\right)(\Xi, \Xi). \tag{14}$$

Hence, if $\Psi \in \ker\left((H_{\lceil \mathcal{D}_0})^* + z^\sharp\right)$, we have $\Psi = 0$ since $\Xi = O^{-1}\Psi = 0$ from (14). Thus the proof is obtained. ■

Here we consider the following condition **(C.2)'**, which is weaker than **(C.2)**.

(C.2)' There exist $e > 0$ such that for $\Phi \in \mathcal{D}(O)$,

$$e\|\Phi\| \ \leq \ \|O\Phi\|.$$

Then the following Proposition 2 holds. It is noted that the conditions **(A)** and **(B)** are stronger than **(ii)** in Proposition A and **(II)** in Theorem 1, respectively.

Proposition 2

Assume **(C.1)**,**(C.2)'** and **(C.3)**. In addition the following **(A)** or **(B)** holds.
(A) There exists a constant $\mu > 0$ such that for all $\Psi \in \mathcal{D}_0$,

$$\mu \ \|O\Psi\|^2 \ \leq \ \left| \ (H\Psi, O\Psi) - (O\Psi, H\Psi) \right| \ . \tag{15}$$

(B) There exists a constant $\nu > 0$ such that for all $\Psi \in \mathcal{D}_0$,

$$\nu \ \|O\Psi\|^2 \ \leq \ \left| \ (H\Psi, O\Psi) + (O\Psi, H\Psi) \right| \ . \tag{16}$$

Then H is essentially self-adjoint on $\mathcal{D}_0$.

(Proof of Proposition 2) Let us prove that $\dim \ker \left(\ (H_{\lceil \mathcal{D}_0})^* + z^\sharp \right) = 0$ for some $z \in \mathbf{C} \backslash \mathbf{R}$. Let $\Psi \in \mathcal{D}((H_{\lceil \mathcal{D}_0})^*)$ and $\Xi = O^{-1}\Psi \ \in \mathcal{D}(O)$. First we suppose **(A)**. Since $\mathcal{D}_0$ is a core of O, it is seen from **(C.3)** and **(A)** that $\mathcal{D}(O) \ \subset \mathcal{D}(\overline{H_{\lceil \mathcal{D}_0}})$ and for $\Phi \in \mathcal{D}(O)$,

$$\mu \ \|O\Phi\|^2 \ \leq \ \left| \ (\overline{H_{\lceil \mathcal{D}_0}}\Phi, O\Phi) - (O\Phi, \overline{H_{\lceil \mathcal{D}_0}}\Phi) \right|. \tag{17}$$

Let $z \in \mathbf{C} \backslash \mathbf{R}$ satisfying $|\mathrm{Im} z| < \frac{\nu e}{2}$. Then, by using $((H_{\lceil \mathcal{D}_0})^*)^* = \overline{H_{\lceil \mathcal{D}_0}}$, we have

$$\left| \mathrm{Im}(\Xi, ((H_{\lceil \mathcal{D}_0})^* + z^\sharp)\Psi) \right| \geq |\mathrm{Im}(\overline{H_{\lceil \mathcal{D}_0}}\Xi, O\Xi)| - |\mathrm{Im} z| \, |(\Xi, O\xi)| \geq \frac{\mu}{2}\|O\Xi\|^2 - |\mathrm{Im} z||(\Xi, O\xi)|. \tag{18}$$

Note that $|(\Xi, O\Xi)| \leq \|\Xi\|\|O\Xi\| \leq \frac{1}{e}\|O\Xi\|^2$ from **(C.2)'**. Then by (18) we have

$$\left| \mathrm{Im}\,(\Xi, ((H_{\lceil \mathcal{D}_0})^* + z^\sharp)\Psi) \right| \geq \left(\frac{\nu}{2} - \frac{|\mathrm{Im} z|}{e} \right) \|O\Xi\|^2 = \left(\frac{\nu}{2} - \frac{|\mathrm{Im} z|}{e} \right) \|\Psi\|^2. \tag{19}$$

Then if $\Psi \in \ker((H_{\lceil \mathcal{D}_0})^* + z^\sharp)$, we have $\Psi = 0$ from (19). Next we suppose **(B)**. Since $\mathcal{D}_0$ is a core of O, it is also seen from **(C.3)** and **(B)** that $\mathcal{D}(O) \ \subset \mathcal{D}(\overline{H_{\lceil \mathcal{D}_0}})$ and for $\Phi \in \mathcal{D}(O)$,

$$\nu \ \|O\Phi\|^2 \ \leq \ \left| \ (\overline{H_{\lceil \mathcal{D}_0}}\Phi, O\Phi) + (O\Phi, \overline{H_{\lceil \mathcal{D}_0}}\Phi) \right| \tag{20}$$

Let $z \in \mathbf{C} \backslash \mathbf{R}$ satisfying $|\mathrm{Re} z| < \frac{\nu e}{2}$ Then, we see that

$$\left| \mathrm{Re}\,(\Xi, ((H_{\lceil \mathcal{D}_0})^* + z^\sharp)\Psi) \right| \geq |\mathrm{Re}(\overline{H_{\lceil \mathcal{D}_0}}\Xi, O\xi)| - |\mathrm{Re} z| \, |(\Xi, O\xi)| \geq \frac{\nu}{2}\|O\Xi\|^2 - |\mathrm{Re} z||(\Xi, O\xi)|. \tag{21}$$

Since $|(\Xi, O\xi)| \leq \|\Xi\| \|O\Xi\| \leq \frac{1}{e}\|O\Xi\|^2$, we have from (21) that

$$\left|\mathrm{Re}(\Xi, ((H_{\lceil \mathcal{D}_0})^* + z^\sharp)\Psi)\right| \geq \left(\frac{\nu}{2} - \frac{|\mathrm{Re}z|}{e}\right)\|O\Xi\|^2 = \left(\frac{\nu}{2} - \frac{|\mathrm{Re}z|}{e}\right)\|\Psi\|^2. \tag{22}$$

Then if $\Psi \in \ker((H_{\lceil \mathcal{D}_0})^* + z^\sharp)$, we have $\Psi = 0$ from (22). Thus the proof is obtained. ■

2 Application

Let us consider the essential self-adjointness of Dirac operators with external potentials. Let us set the state space by $\mathcal{H}_\mathrm{D} = \mathbf{C}^4 \otimes L^2(\mathbf{R}^3)$, and the free Hamiltonian by

$$H_0 \;=\; \sum_{j=1}^{3} \alpha^j \otimes \hat{P}_j \;+\; \beta \otimes M, \qquad\qquad M > 0,$$

where $\hat{P}_j = -i\partial_{x^j}$, and α^j, $j = 1,2,3$, β are 4×4 hermitian matrices satisfying $\{\alpha^j, \alpha^l\} = 2\delta_{j,l}$, $\{\alpha^j, \beta\} = 0$, and $\beta^2 = I$. It is seen that H_0 is self-adjoint on $\mathcal{D}(H_0) = \cap_{j=1}^3 \mathcal{D}(\hat{P}_j)$, and essentially self-adjoint on $\mathcal{D}_\mathrm{c} = \mathbf{C}^4 \otimes C_0^\infty(\mathbf{R}^3)$. From the functional calculus we see that

$$M\|\Psi\| \;\leq\; \|H_0\Psi\|, \qquad\qquad \Psi \in \mathcal{D}(H_0). \tag{23}$$

Let us set the total Hamiltonian by

$$H(\kappa) \;=\; H_0 + \kappa V, \qquad\qquad \kappa \in \mathbf{R},$$

where V is a symmetric operator satisfying the condition below :

(D.1) $\mathcal{D}_\mathrm{c} \subset \mathcal{D}(V)$, and there exist $a > 0$ and $b \geq 0$ such that for $\Psi \in \mathcal{D}_\mathrm{c}$,

$$\|V\Psi\| \;\leq\; a\|H_0\Psi\| \;+\; b\|\Psi\|.$$

Then, the Kato-Rellich theorem shows that for $|\kappa| < \frac{1}{a}$, $H(\kappa)$ is self-adjoint on $\mathcal{D}(H_0)$, and essentially self-adjoint on $\mathcal{D}_\mathrm{c}$. Here we consider another proof by applying Proposition 2. We assume the additional condition :

(D.2) There exists $c > 0$ such that for $\Psi \in \mathcal{D}_\mathrm{c}$,

$$\left|(H_0\Psi, V\Psi) + (V\Psi, H_0\Psi)\right| \;\leq\; c\,\|H_0\Psi\|^2.$$

Then we obtain the corollary below :

Corollary 3

Assume **(D.1)** and **(D.2)**. Then for $|\kappa| < \frac{2}{c}$, $H(\kappa)$ is essentially self-adjoint on $\mathcal{D}_{\rm c}$.

(Proof) Let us apply $H(\kappa)$ to H and $H_{\rm D}$ to O in Proposition 2. We see that **(C.1)** follows, and **(C.2)'** holds from (23). By **(D.1)** and (23), we have $\|H(\kappa)\Psi\| \leq (1+|\kappa|a)\|H_0\Psi\| + b\|\Psi\|$ for $\Psi \in \mathcal{D}_{\rm c}$. Thus **(C.3)** follows. We also see from **(D.2)** that for $\Psi \in \mathcal{D}_{\rm c}$,

$$|(H(\kappa)\Psi, H_0\Psi) + (H_0\Psi, H(\kappa)\Psi)| \geq 2\|H_0\Psi\|^2 - |\kappa| \left((V\Psi, H_0\Psi) + (H_0\Psi, V\Psi) \right) | \geq (2-|\kappa|c)\|H_0\Psi\|^2.$$

Then the condition **(B)** in Proposition 2 is satisfied. Thus we obtain the corollary. ∎

Let us consider an example of V in Corollary 3. Let us set

$$V \ = \ \tau \otimes K$$

where $\tau = \beta\alpha^1\alpha^2\alpha^3$, and K is the symmetric operator satisfying the conditions below :

(E.1) $C_0^\infty(\mathbf{R}^3) \subset \mathcal{D}(K)$, and there exist $a_1 \geq 0$ and $b_1 \geq 0$ such that

$$\|K\psi\|^2 \ \leq \ a_1 \sum_{j=1}^3 \|\hat{P}_j\psi\|^2 \, + \, b_1\|\psi\|^2, \qquad \psi \in C_0^\infty(\mathbf{R}^3).$$

(E.2) $K\left(C_0^\infty(\mathbf{R}^3)\right) \subset C_0^\infty(\mathbf{R}^3)$, and there exist $a_2 \geq 0$ and $b_2 \geq 0$ such that

$$\sum_{j=1}^3 \|[K, \hat{P}_j]\psi\|^2 \leq a_2 \sum_{j=1}^3 \|\hat{P}_j\psi\|^2 + b_2\|\psi\|^2, \qquad \psi \in C_0^\infty(\mathbf{R}^3).$$

For an example of K, we take $K = P(\hat{P}_1, \hat{P}_2, \hat{P}_3)$ where P is a polynomial of order one. Now let us show that V satisfies **(D.1)-(D.2)**. Note that τ is hermitian and $\tau^2 = I$. Then by **(E.1)**, we have

$$\|H(\kappa)\Psi\| \leq \|H_0\Psi\| + |\kappa|\|I \otimes K\Psi\| \ \leq \left(1+|\kappa|a_1^{1/2}\right)\|H_0\Psi\| + b_1^{1/2}\|\Psi\|, \tag{24}$$

for $\Psi \in \mathcal{D}_{\rm c}$. Thus **(D.1)** follows. Note that $\{\tau, \beta\} = 0$ and $\{\tau, \alpha^j\} = 0$, $j = 1, 2, 3$. Then by using $\{X_1 \otimes Y_1, X_2 \otimes Y_2\} = \{X_1, X_2\} \otimes Y_1Y_2 + X_2X_1 \otimes [Y_2, Y_1]$, we have

$$\left| ((\tau \otimes K)\Psi, H_0\Psi) + (H_0\Psi, (\tau \otimes K)\Psi) \right| \ = \ \left| \sum_{j=1}^3 (\Psi, ((\alpha^j\tau) \otimes [\hat{P}_j, V_0])\Psi)) \right| \leq \ \|\Psi\| \sum_{j=1}^3 \|[V_0, \hat{P}_j]\Psi\| \tag{25}$$

Then by (23) and **(E.2)**, we have

$$\left| ((\tau \otimes K)\Psi, H_0\Psi) + (H_0\Psi, (\tau \otimes K)\Psi) \right| \ \leq \ \frac{1}{M}\left(a_2 + \frac{b_2}{M^2}\right)^{1/2} \|H_{\rm D}\Psi\|^2. \tag{26}$$

Then from corollary 3, for $|\kappa| < \frac{2M}{\sqrt{a_2+\frac{b_2}{M^2}}}$, $H_{\rm D}(\kappa)$ is essentially self-adjoint on $D_{\rm c}$.

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